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Semistable principal bundles—I (characteristic zero)

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Dedicated to Claudio Procesi on the occasion of his 60th birthday

Introduction

The moduli space of principal G-bundles (for a reductive algebraic group G) on a smooth projective curve X was constructed by A. Ramanathan over fields of characteristic zero (cf. [R1,R2]). His method using Geometric Invariant Theory followed the basic lines of the construction for the case of vector bundles (cf. [Ses]). The properness (and hence the projectivity) of the moduli space is an end product of this method of construction. One knows that this property (for the case of vector bundles) could be proved a priori, before constructing the moduli spaces and is referred to as the *semistable reduction theorem* (cf. Langton [L]).

The principal aim in this article is to prove this semistable reduction theorem for principal G-bundles over X in characteristic zero (cf. Theorem 7.1); that is, the moduli functor associated to semistable principal G-bundles is proper. The construction of these moduli spaces follows as an easy consequence from the case of vector bundles (cf. Section 8).

Our approach could be termed *Tannakian* in the sense that a G-bundle can be viewed as a *tensor functor* and can be studied in terms of its associated vector bundles. This arose out of an attempt to understand C. Simpson's proof of results

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similar to Lemma 8.1, and Lemma 8.2 which he proves in the context of Higgs bundles by using *Tannakian arguments* (cf. [Sim2,BBN]).

The techniques developed in the proof of Theorem 7.1 are general and have many applications. For instance, we prove the semistable reduction theorem for families of semistable principal Higgs bundles over smooth projective varieties (cf. Theorem 9.3). In particular, we get a different proof of Theorem 9.15 in [Sim2].

The most important application is that the methods of this paper generalize suitably to fields of positive characteristic as well and this appears in a sequel to this paper (cf. [BP]).

There is also a proof, due to G. Faltings, of the semistable reduction theorem for principal G-bundles in char 0 (cf. [F]). In an earlier article of ours [Rem], the proof of this theorem had a serious error which was pointed out by G. Faltings.

Since the proof of the semistable reduction theorem is technically involved we outline the broad strategy so as to highlight the main difference between the present approach and the existing ones. This would enable the reader to appreciate how this method is amenable for generalization to positive characteristics (see [BP]).

0.1. Outline of proof of the semistable reduction theorem

The notations are as in Section 1, where A is a dvr with residue field k, which is algebraically closed and the function field of A is K. We are given a family of semistable principal H_K-bundles on X_K. The problem is to extend this as a semistable H_A-bundle to X_A.

We choose a faithful representation $H \hookrightarrow G$, where $G = SL(n)$. Extending the structure group of P_K to G_K over X_K we call this G_K-bundle as E_K. Then, by using the GIT construction of the moduli space of vector bundles, we extend this to a G_A-bundle E_A on X_A with the added property that the limiting bundle is *polystable*. (For this we may need to go to a finite cover of A.)

We now view the entire data given above as follows: we are given E_A, a G_A-bundle on X_A, together with a reduction of structure group to H_K over X_K. The reduction gives a section

$$s_K : X_K \to E_K(G_K/H_K).$$

The point is that, if this holds, the semistable reduction theorem follows. One of the crucial technical results, namely, Proposition 2.8 is that, if this section s_K extends along any point $x \in X$ i.e. along $x_A = x \times \operatorname{Spec}(A)$, to a section of $E_A(G_A/H_A)|x_A$, then the semistability of the family E_A enables us to prove that s_K extends to s_A.

The difficulty is that s_K need not extend along any $x \in X$. One attempts to get around this as follows: we fix a base point $x \in X$; we also fix a non-canonical A-section of $E_A|x_A$. Given this, the reduction section s_K along x_K can be thought of as giving a *coset representative* $\theta_K.H_K$ in G_K/H_K, which is not in general extendable to a *coset* $\theta_A.H_A$.

One modifies the group scheme to a conjugate group scheme $H'_K = \theta_K.H_K.\theta_K^{-1}$ and views s_K as a section of $E_K(G_K/H'_K)$. Then it is a simple observation that, s_K restricted to x_K is indeed the *identity coset* $e_K.H'_K$ in G_K/H'_K. Going to the *flat closure* H'_A of H'_K in G_A, this extends as the *identity coset* of G_A/H'_A. Observe that the group scheme H'_A need not be semisimple.

Viewed thus, the section s_K extends along the base section x_A to a section of $E_A(G_A/H'_A)$, where H'_A is the flat closure of H'_K in G_A. Thus, *the gain of extending the reduction section along x_A forces the choice of the flat closure in the category of non-semisimple group schemes*. The key points of the proof thereafter are the following:

1. Extend the section s_K to a section s_A of $E_A(G_A/H'_A)$ over the whole of X_A. In other words, reduce the structure group of the G_A-bundle E_A to the flat closure H'_A (cf. Section 5, Proposition 5.1).
2. Using Bruhat–Tits theory, relate the group schemes H'_A and H_A so as to obtain the required H_A-bundle (cf. Section 6, Proposition 6.2).

It is probably appropriate at this juncture to observe the basic difference between this proof and Langton's proof in the case of families of vector bundles.

In his proof, Langton first extends the family of semistable vector bundles (or equivalently principal GL_n-bundles) to a GL_n-bundle in the limit although non-semistable. In other words, the structure group of the limiting bundle remains GL_n. Then by a sequence of *Hecke modifications* he reaches the semistable limit without changing the isomorphism class of the bundle over the generic fiber.

Instead, we extend the family of semistable H_K-bundles to an H'_A-bundle with the limiting bundle remaining semistable, but the structure group is non-reductive in the limit. In other words one loses the reductivity of the structure group scheme. Then, by using Bruhat–Tits theory (cf. Definition 3.2), we relate the group scheme H'_A to the reductive group scheme H_A without changing the isomorphism class of the bundle over the generic fiber as well as the semistability of the limiting bundle.

The layout of the paper is as follows: Section 2 contains some preliminary results on principal bundles which are crucial for what follows. Section 3 to Section 7 is devoted to the proof of the semistable reduction theorem; Section 8 gives the construction of the moduli space of semistable principal bundles. In Section 9 we indicate briefly how the methods in Sections 3–7 extend to the case of principal Higgs bundles.

1. Notations and conventions

Throughout this paper, unless otherwise stated, we have the following notations and assumptions:

(a) We work over an algebraically closed field k of characteristic zero and without loss of generality we can take k to be the field of complex numbers $\mathbb{C}$.
(b) H is a *semisimple* algebraic group, and G, unless otherwise stated will always stand for the general linear group $GL(n)$. Their representations are finite dimensional and rational.
(c) X is a smooth projective curve of genus $g \geqslant 2$.
(d) A is a discrete valuation ring (which could be assumed to be complete) with residue field k, and quotient field K.
(e) Let E be a principal G-bundle on $X \times T$ where T is $\operatorname{Spec} A$. Let $x \in X$ be a closed point which we fix throughout. Then throughout this article we shall denote by $E_{x,A}$ or $E_{x,T}$ (respectively $E_{x,K}$) the restriction of E to the subscheme $x \times \operatorname{Spec} A$ or $x \times T$ (respectively $x \times \operatorname{Spec} K$). Similarly, $p \in T$ will denote the closed point of T and the restriction of E to $X \times p$ will be denoted by E_p.
(f) In the case of $G = GL(n)$, when we speak of a principal G-bundle we identify it often with the associated vector bundle (and can therefore talk of the degree of the principal G-bundle).
(g) We denote by E_K (respectively E_A) the principal bundle E on $X \times \operatorname{Spec} K$ (respectively $X \times \operatorname{Spec} A$) when viewed as a principal H_K-bundle (respectively H_A-bundle). Here H_K and G_K (respectively H_A and G_A) are the product group scheme $H \times \operatorname{Spec} K$ and $G \times \operatorname{Spec} K$ (respectively $H \times \operatorname{Spec} A$ and $G \times \operatorname{Spec} A$).
(h) If H_A is an A-group scheme, then by $H_A(A)$ (respectively $H_K(K)$) we mean its A (respectively K)-valued points. When $H_A = H \times \operatorname{Spec} A$, then we simply write $H(A)$ for its A-valued points. We denote the closed fiber of the group scheme by H_k.
(i) Let Y be any G-module and let E be a G-principal bundle. For example Y could be a G-module. Then we denote by $E(Y)$ the associated bundle with fiber type Y which is the following object: $E(Y) = (E \times Y)/G$ for the twisted action of G on $E \times Y$ given by $g.(e, y) = (e.g, g^{-1}.y)$.
(j) If we have a group scheme H_A (respectively H_K) over $\operatorname{Spec} A$ (respectively $\operatorname{Spec} K$) an H_A-module Y_A and a principal H_A-bundle E_A. Then we shall denote the associated bundle with fiber type Y_A by $E_A(Y_A)$.
(k) By a family of H-bundles on X parameterized by T we mean a principal H-bundle on $X \times T$, which we also denote by $\{E_t\}_{t \in T}$.

2. Preliminaries

Remark 2.1. Recall that if $H \subset G$

(a) a principal G-bundle E on X is said to have an H-structure or equivalently a reduction of structure group to H if we are given a section $\sigma : X \to E(G/H)$, where $E(G/H) \simeq E \times^G G/H$;

(b) further, there is a natural action of the group $\mathrm{Aut}_G E$, of automorphisms of the principal G-bundle E, on $\Gamma(X, E(G/H))$ and the orbits correspond to the H-reductions which are isomorphic as principal H-bundles.

Lemma 2.2. *Let V and W be semistable vector bundles on X of degree zero. Then $V \otimes W$ is semistable of degree zero.*

Proof. Any semistable bundle on X of degree zero has a filtration such that its associated graded is a direct sum of stable bundles of degree zero. Hence the tensor product $V \otimes W$ gets a filtration such that its associated graded is a direct sum of tensor products of stable bundles of degree zero. We see easily that this reduces to proving the lemma when V and W are stable of degree zero. Then by the Narasimhan–Seshadri theorem, $V \otimes W$ is defined by a unitary representation of the fundamental group (namely the tensor product of the irreducible unitary representations which define V and W, respectively), which implies that $V \otimes W$ is semistable (cf. [Ses]).

Proposition 2.3. *Let E be a principal H-bundle on X. Then the following are equivalent*:

(a) *There exists a faithful representation $H \hookrightarrow GL(V)$ such that the induced bundle $E(V) = E \times^H V$ is semistable* (*of degree zero*).
(b) *For every representation $H \to GL(W)$, the bundle $E(W)$ is semistable* (*of degree zero*).

Proof. (b) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (b). Since H is semisimple; the vector bundle $E(V)$ is semistable of degree 0. Consider the natural tensor representation $T^{a,b}(V) = \bigotimes^a V \otimes \bigotimes^b V^*$. Then by Lemma 2.2, the bundle $E(T^{a,b}(V)) = \bigotimes^a E(V) \otimes \bigotimes^b E(V)^*$ is semistable of degree 0.

It is well-known that any H-module W is a subquotient of a suitable $T^{a,b}(V)$ and hence $E(W)$ is a subquotient of $E(T^{a,b}(V))$ of degree zero. Therefore $E(W)$ is also semistable.

Definition 2.4. An H-bundle E is said to be *semistable* if it satisfies the equivalent conditions in Proposition 2.3.

Definition 2.5. Let H' be an affine algebraic group not necessarily reductive. Let P be a principal H'-bundle on X. We define P to be *semistable* if it is *flat* (in the sense that it comes from the representation of the fundamental group of X) and there exists a faithful representation

$$\rho : H' \to GL(V)$$

such that the associated vector bundle $P(V)$ is *semistable of degree zero*.

Proposition 2.6. *Let H' be an affine algebraic group not necessarily reductive as above and let P be a semistable principal H'-bundle. Let $f : H' \to H$ be a morphism from H' to a semisimple group H. Then the associated principal H-bundle $P(H)$ is also semistable.*

Proof. By Proposition 2.3 we need only check that if $\psi : H \to GL(W)$ is any representation of H then the associated bundle $P(H)(W)$ is semistable. In other words, we need to check that if $\gamma : H' \to GL(W)$ (e.g., $\gamma = \psi \circ f$) is any representation of H', not necessarily faithful, then the bundle $P(W)$ is semistable.

Observe that by Definition 2.5 we have a faithful representation $GL(V)$ of H' such that $P(V)$ is semistable of degree zero. Further, we are over a field of characteristic zero and so the H'-module W can be realized as a subquotient of a direct sums of some $T^{a,b}(V)$ (cf., for example, [Sim0, p. 86]). Hence the vector bundle $P(W)$ is a subquotient of $\bigoplus T^{a,b}(P(V))$. Since $P(V)$ is semistable of degree zero, so is $T^{a,b}(P(V))$.

Since P is *flat* the associated vector bundles of all the subquotients of the tensor representations are of *degree zero*. Again subquotients of semistable vector bundles of degree zero are semistable. Hence $P(W)$ is semistable of degree zero, which proves the proposition.

Remark 2.7. See also Definition 8.7 for the intrinsic definition of semistability of principal bundles due to A. Ramanathan.

Let G be $GL(n)$ and let H be a semisimple algebraic group, $H \subset G$. Let

$$F_G : (\mathit{Schemes}) \to (\mathit{Sets})$$

be the functor given by

$$F_G(T) = \left\{ \begin{array}{l} \text{isomorphism classes of semistable } G\text{-bundles of degree } 0 \\ \text{on } X \text{ parameterised by } T \end{array} \right\}.$$

One may similarly define the functor F_H (note that since H is semisimple, for a principal H-bundle the associated vector bundles have degree zero).

Let $x \in X$ be a marked point and let $F_{H,G,x}$ be the functor

$$F_{H,G,x}(T) = \left\{ \begin{array}{l} \text{isomorphism classes of pairs } (E, \sigma_x)\text{: } E = \{E_t\}_{t \in T} \text{ is} \\ \text{a family of semistable principal } G\text{-bundles of degree } 0 \\ \text{and } \sigma_x : T \to E(G/H)_x \text{ is a section} \end{array} \right\}.$$

(Recall that $E(G/H)_x$ denotes the restriction of $E(G/H)$ to $x \times T \approx T$.)

Notice that the functor F_H is in fact realizable as the following functor (by Remark 2.1(a)):

$$F_{H,G}(T) = \left\{ \begin{array}{l} \text{isomorphism classes of pairs } (E, s)\text{: } E = \{E_t\}_{t \in T} \text{ is} \\ \text{a family of } G\text{-bundles of degree } 0 \text{ and} \\ s = \{s_t\}_{t \in T} \text{ is a section of } E(G/H) \text{ on } x \times T \\ \text{or what we may call a family of sections of } \{E(G/H)_t\}_{t \in T} \end{array} \right\}.$$

In what follows, we shall identify the functors F_H with $F_{H,G}$. With these definitions we have the following proposition.

Proposition 2.8. *Let α_x be the morphism induced by "evaluation of section" at x:*

$$\alpha_x : F_H \to F_{H,G,x}.$$

Then α_x is a proper morphism of functors (*cf.* [DM]).

Proof. Let T be an affine smooth curve and let $p \in T$. Then by the valuation criterion for properness, we need to show the following:

> If E is a family of semistable principal G-bundles on $X \times T$ together with a section $\sigma_x : T \to E(G/H)_x$ such that for $t \in T - p$, we are given a family of H-reductions, i.e. a family of sections $s_{T-p} = \{s_t\}_{t \in T-p}$, where $s_t : X \to E(G/H)_t$ has the property that, at x, $s_t(x) = \sigma_x(t)$ $\forall t \in T - p$;

then we need to extend the family s_{T-p} to a section s_T of $E(G/H)$ on $X \times T$ so that $s_p(x) = \sigma_x(p)$ as well.

Observe that since G/H is affine, there exists a G-module W such that $G/H \overset{i}{\hookrightarrow} W$ is a *closed G-embedding*. Thus we get a closed embedding

$$E(G/H) \hookrightarrow E(W)$$

and a family of semistable vector bundles $\{E(W)_t\}_{t \in T}$ together with a family of sections s_{T-p} and evaluations $\{\sigma_x(t)\}_{t \in T}$ such that $s_t(x) = \sigma_x(t)$, $t \neq p$.

For the section s_{T-p}, viewed as a section of $E(W)_{T-p}$ we have two possibilities:

(a) it extends as a regular section s_T;
(b) it has a pole along $X \times p$.

Observe that if (a) holds, then we have

$$s_T\big(X \times (T - p)\big) \subset E(G/H) \subset E(W),$$

since $E(G/H)$ is closed in $E(W)$, it follows that $s_T(X \times p) \subset E(G/H)$. Thus $s_p(X) \subset E(G/H)_p$. Further, by continuity, $s_p(x) = \sigma_x(p)$ as well, and this proves the proposition.

To complete the proof, we need to check that the possibility (b) cannot hold.

Suppose it does hold. For our purposes, we could take the local ring A of T at p, which is a discrete valuation ring with a uniformizer π. Let K be its quotient field. Then the section $s_{T-p} = s_K$ is a section of $E(W)_K$; in other words, a *rational section* of $E(W)$ with a pole along the divisor $X \times p \subset X \times T$ of order $k \geqslant 1$. Thus, by multiplying s_{T-p} by π^k we get a regular section s'_T of $E(W)$ on $X \times T$. If $s'_T = \{s'_t\}_{t \in T}$, then we have:

(i) $s'_t = \lambda(t) \cdot s_t$, $t \in T - p$, where $\lambda : T \to \mathbb{C}$ is a function given by π^k, having zeros of order k at p.

(ii) s'_p is a non-zero section of $E(W)_p$. Notice that by (ii), since s'_p is a section of $E(W)_p$ and $E(W)_p$ is a *semistable bundle* of degree 0, a non-vanishing section is nowhere vanishing, i.e.,

$$s'_p(y) \neq 0 \quad \forall y \in X. \tag{$*$}$$

By assumption, $s_t(x) = \sigma_x(t)$, $t \in T - p$, hence

$$s'_t(x) = \lambda(t) \cdot \sigma_x(t), \quad t \in T - p.$$

Therefore, by continuity, since $\sigma_x(p)$ is well-defined, we see that $\lambda(t) \cdot \sigma_x(t)$ tends to $\lambda(p) \cdot \sigma_x(p) = 0$ as $t \to p$.

Also $s'_t(x) \to s'_p(x)$ as $t \to p$. Hence, by continuity, it follows that $s'_p(x) = 0$, which contradicts $(*)$.

Thus the possibility (b) does not occur and we are done.

Remark 2.9. For a different proof of this proposition see [BP, Proposition 3.12].

We isolate the above proof for future use in the following lemma.

Lemma 2.10. *Let $T = \operatorname{Spec} A$ and let E_T be a family of semistable vector bundles of degree zero on $X \times T$. Let s_K be a section of the family E_K restricted to $X \times \operatorname{Spec} K$, with the property that for a base point $x \in X$, the section s_K extends along $x \times T$ to give a section of $E_{x \times T}$. Then the section extends to the whole $X \times T$.*

3. Towards the flat closure

Fix a faithful representation $H \hookrightarrow G$ defined over $\mathbb{C}$. Consider the extension of structure group of the bundle P_K via the induced K-inclusion $H_K \hookrightarrow G_K$. We denote the associated G_K-bundle $P_K(G)$ by E_K.

Then, since $G = GL(n)$, by the properness of the moduli space of semistable vector bundles, there exists a *semistable extension* of $P_K(G) = E_K$ to a G_A-bundle on $X \times \operatorname{Spec} A$, which we denote by E_A. Call the restriction of E_A to $X \times p$ (identified with X) the *limiting bundle* of E_A and denote it by E_p (as in Section 1). One has in fact slightly more, which is what we need.

Lemma 3.1. *Let E_K denote a family of semistable G_K-bundles of degree zero on $X \times \operatorname{Spec} K$ (or equivalently a family of semistable vector bundles of rank n and degree zero on $X \times (T - p)$). Then, (by going to a finite cover S of T if need be) the principal bundle E_K extends to E_A with the property that the limiting bundle E_p is in fact polystable, i.e., a direct sum of stable bundles of degree zero.*

Proof. This lemma is quite standard but we shall prove it in Section 4.

3.1. The flat closure

We observe the following:

- Note that giving the H_K-bundle P_K is giving a reduction of structure group of the G_K-bundle E_K which is equivalent to giving a section s_K of $E_K(G_K/H_K)$ over X_K.
- We fix a base point $x \in X$ and denote by $x_A = x \times \operatorname{Spec} A$, the induced section of the family (which we call the *base section*):

  $$X_A \to \operatorname{Spec} A.$$

- Let $E_{x,A}$ (respectively $E_{x,K}$) be as in Section 1, the restriction of E_A to x_A (respectively x_K). Thus, $s_K(x)$ is a section of $E_K(G_K/H_K)_x$ which we denote by $E_x(G_K/H_K)$.
- Since $E_{x,A}$ is a principal G-bundle on $\operatorname{Spec} A$ and therefore trivial, it can be identified with the group scheme G_A itself. *For the rest of the article we fix one such identification, namely*:

  $$\xi_A : E_{x,A} \to G_A.$$

- Since we have fixed ξ_A, we have a canonical identification

  $$E_x(G_K/H_K) \simeq G_K/H_K,$$

  which therefore carries a natural *identity section* e_K (i.e. the coset $\mathrm{id}.H_K$). Using this identification we can view $s_K(x)$ as an element in the homogeneous space G_K/H_K.
- Let $\theta_K \in G(K)$ be such that $\theta_K^{-1} \cdot s_K(x) = e_K$ (for this we may have to go to a finite extension of K). Then we observe that, the isotropy subgroup scheme in G_K of the section $s_K(x)$ is $\theta_K.H_K.\theta_K^{-1}$.
- On the other hand, one can realize $s_K(x)$ as the identity coset of $\theta_K.H_K.\theta_K^{-1}$ by using the following identification:

  $$G_K/\theta_K.H_K.\theta_K^{-1} \xrightarrow{\sim} G_K/H_K, \qquad g_K\big(\theta_K.H_K.\theta_K^{-1}\big) \mapsto g_K\theta_K.H_K.$$

Definition 3.2. Let H'_K be the subgroup scheme of G_K defined as

$$H'_K := \theta_K.H_K.\theta_K^{-1}.$$

Using ξ_A we can have a canonical identification

$$E_x\big(G_K/H'_K\big) \simeq G_K/H'_K.$$

Then, we observe that, using the above identification we get a section s'_K of $E_K(G_K/H'_K)$, with the property that, $s'_K(x)$ is the *identity section* and moreover, since we have conjugated by an element $\theta_K \in G_A(K)(= G(K))$, the isomorphism class of the H_K-bundle P_K given by s_K does not change by going to s'_K.

Thus, in conclusion, the G_A-bundle E_A has a reduction to H'_K given by a section s'_K of $E_K(G_K/H'_K)$, with the property that, at the given base section $x_A = x \times \operatorname{Spec} A$, we have an equality $s'_K(x_A) = e'_K$ with the *identity element* of G_K/H'_K (namely the coset id.H'_K).

Definition 3.3. Let H'_A be the *flat-closure* of H'_K in G_A.

We then have a canonical identification via ξ_A:

$$E_x\big(G_A/H'_A\big) \simeq G_A/H'_A.$$

By definition, since H'_K is reduced, H'_A is the scheme theoretic closure of H'_K in G_A with the canonical *reduced* structure. One can easily check that H'_A is indeed a subgroup scheme of G_A since it contains the identity section of G_A, and moreover, it is faithfully flat over A. However, notice that H'_A *need not* be a *reductive* group scheme; that is, the special fiber H_k over p need not be reductive.

Observe further that, $s'_K(x)$ extends in a trivial fashion to a section $s'_A(x)$; namely, the *identity coset section* e'_A of $E_x(G_A/H'_A)$ identified with G_A/H'_A.

Remark 3.4. If H'_A is *reductive* then the semistable reduction theorem follows quite easily. For, firstly, by the *rigidity* of reductive group schemes over $\operatorname{Spec} A$ [SGA3, Expose III, Corollary 2.6, p. 117], by going to a finite cover, we may assume that $H'_A = H \times \operatorname{Spec} A$. Then we have a *closed G-immersion* of G/H in a G-module W, and one may view s_K as a section of $E_K(W_K) \hookleftarrow E_K(G_K/H'_K)$. By choice, along x_A, the section $s_K(x)$ extends regularly to a section of $E_A(G_A/H'_A) \subset E_A(W_A)$. Hence, by Proposition 2.8, s_K extends to a section s_A, which gives the required reduction over $X \times \operatorname{Spec} A$.

4. Chevalley embedding of G_A/H'_A

As we have noted, H'_A need not be reductive and the rest of the proof is to get around this difficulty. Our first aim is to prove that the structure group of the bundle $E_A(G_A)$ can be reduced to H'_A, which is the statement of Theorem 5.1.

We need to prove the following generalization of a well-known result of Chevalley.

Lemma 4.1. *There exists a finite dimensional G_A-module W_A such that $G_A/H'_A \hookrightarrow W_A$ is a G_A-immersion.*

Proof. We follow Chevalley's proof. Let I_K be the ideal defining the subgroup scheme H'_K in $K(G)$ (note that G_A (respectively G_K) is an affine group scheme and we denote by $A(G)$ (respectively $K(G)$) its coordinate ring).

Set $I_A = I_K \cap A(G)$. Then it is easy to see that since we are over a discrete valuation ring, I_A is in fact the ideal in $A(G)$ defining the flat closure H'_A. Observe also that I_A is a *primitive* A submodule of $A(G)$, that is, $A(G)/I_A$ is torsion free; further, $I_A \otimes k = I_k$ is the defining ideal in $k(G)$ of H'_k in G_k and $I_A \otimes K$ is I_K. We may now choose a finite generating set $\{f_i\}$ of I_K, such that modulo k, their images $f_{i,k}$ generate I_k.

As in the classical proof of Chevalley, one has a finite dimensional G_K-submodule, V_K, containing the $\{f_i\}$. Now set $V_A = V_K \cap A(G)$ and $M = V_A \cap I_A$. Observe that I_A, V_A and hence M are all G_A-submodules of $A(G)$. This can be seen by keeping track of the comodule operations. Then clearly V_A is primitive in $A(G)$ and M is also primitive in $A(G)$ and in particular, primitive in V_A. If we set

$$M_k = M \otimes k \quad \text{and} \quad V_k = V_A \otimes k,$$

we see that the inclusion $M \hookrightarrow V_A$ induces an inclusion $M_k \hookrightarrow V_k$. Observe that

$$f_i \in M, \qquad f_{i,k} \in M_k, \quad \text{and} \quad M \subset I_A,$$

$$M_k \subset I_k \quad \text{and} \quad M_k = V_k \cap I_k.$$

We claim that, for $g \in G_A(k)$, one has

$$g \cdot M_k \subset M_k \quad \Longleftrightarrow \quad g \in H'_k.$$

Obviously, if $g \in H'_k$, then $g \cdot M_k \subset M_k$, since V_k is G-stable and I_k is H'_k-stable. Thus it suffices to show that

$$f_{i,k}(g) = 0 \quad \text{for all } i;$$

that is, $f_{i,k}$ vanish on g. Since $f_{i,k} \in M_k$, it suffices to show that

$$F(g) = 0 \quad \text{for } F \in M_k.$$

But $F(g) = (g^{-1} \cdot F)(\mathrm{id})$, where $g^{-1} \cdot F$ is the action of G on functions on G. Now, by hypothesis, $(g^{-1} \cdot F) \in M_k$. Since $M_k \subset I_k$, and $\mathrm{id} \in H'_k$, we see that $(g^{-1} \cdot F)(\mathrm{id}) = 0$. This proves the above claim. □

Similarly, if we set

$$M_F = M \otimes_A L \quad \text{and} \quad V_F = V_A \otimes_A F,$$

where F is any field containing A, we see that for $g \in G(F)$

$$g \cdot M_F \subset M_F \quad \Longleftrightarrow \quad g \in H'_A(F).$$

Let L denote the primitive rank-one A-submodule $\bigwedge^d M \hookrightarrow \bigwedge^d V = W_A$, and [L] the A-valued point of $\mathbb{P}(W_A)$ defined by L. Here, $\mathbb{P}(W_A)$ is defined by the functor associated to rank-one direct summands of W_A. Then, the above discussion means that, we can recover H'_A as the isotropy subgroup scheme at [L] for the G_A-action on $\mathbb{P}(W_A)$.

Recall that, for any field F, the isotropy subgroup of $G_A(F)$, at the point of $\mathbb{P}W_A(F)$ represented by the base change of L by F, is $H'_A(F)$.

Fix a generator $l \in L$ so that l is a primitive element in W_A and consider the isotropy subgroup scheme H''_A at l for the G_A-action on W_A. We claim that, H''_A coincides with H'_A. To see this, observe that, H''_A is the subgroup scheme of G_A which leaves the closed subscheme ($= \mathrm{Spec}(A)$) determined by l invariant (with the corresponding automorphism on this subscheme being identity). We see then that, H''_A is a *closed* subgroup scheme of G_A. Further, we see that $H''_A \hookrightarrow H'_A$. Since H'_K is semi-simple, it has no characters and therefore the isotropy subgroup scheme at $(l \otimes K) \in (W_A \otimes K)$ is precisely H'_K. This means that $H''_K = H'_K$. Now, H'_K is open (dense) in H'_A (since H'_A is the flat closure of H'_K in G_A) so that H'_K is also dense in H''_A. This implies that H'_A and H''_A coincide set-theoretically. Observe also that H'_A is *reduced* by the definition of flat closure. Thus, it follows that $H'_A = H''_A$. This implies that $G_A/H'_A \hookrightarrow W_A$ is a G_A*-immersion*, and the above lemma follows.

Remark 4.2. Regarding the Lemma 4.1 proved above, we note that usually the subgroup scheme H'_A can be realized only as the isotropy subgroup scheme of a line in a G_A-module. But here, since the generic fiber of H'_A is semisimple, one is able to realize H'_A as the isotropy subgroup scheme of a primitive element in a G_A-module and the limiting group also as an isotropy subgroup scheme for an element in a G_k-module.

5. Extension to flat closure and local constancy

Recall that the section $s'_K(x)$ extends along the base section x_A to give $s'_A(x) = w_A$. The aim of this section is to prove the following key theorem.

Theorem 5.1. *The section s'_K extends, in fact, to a section s'_A of $E_A(G_A/H'_A)$. In other words, the structure group of E_A can be reduced to H'_A; in particular, if H'_k denotes the closed fiber of H'_A then the structure group of E_k can be reduced to H'_k.*

For the proof we need the following key result.

Proposition 5.2. *Let E be a polystable principal G-bundle on X of degree zero (here $G = SL(n)$ or $GL(n)$). Let W be a G-module and Y a G-subscheme of W of the form G/H' where $H' = \mathrm{Stab}_G(w)$ for some $w \in W$.*

Let s be a section of $E(W)$ such that for some $x \in X$, $s(x) = w$ in the fiber of $E(Y)$ at $x \in X$. Then the entire image of s lies in $E(Y)$. In particular, E has a reduction of structure group to the subgroup H'. Furthermore, this reduction of structure group to H' is flat, in the sense that the H'-bundle comes from the representation of the fundamental group $\pi_1(X)$ of X. In other words, the reduced H'-bundles is semistable of degree zero *in the sense of Proposition* 2.6.

Proof. The bundle E being polystable, it is defined by a unitary representation

$$\chi : \pi_1(X) \to G$$

which maps into the unitary subgroup of G. This implies that if the universal covering $j : Z \to X$ is considered as a principal fiber space with structure group $\pi_1(X)$, then the principal G-bundle E is the associated bundle through χ.

Let $\rho : G \to GL(W)$ be the representation defining the G-module W. Then $E(W)$ can be considered as the bundle associated to the principal bundle $j : Z \to X$ through the representation

$$\rho \circ \chi : \pi_1(X) \to GL(W),$$

which maps into the unitary subgroup of $GL(W)$.

By generalities on principal bundles and associated constructions, since

$$E(W) \simeq Z \times^{\pi_1(X)} W,$$

a section of $E(W)$ can be viewed as a $\pi_1(X)$-map $s_1 : Z \to W$. Now, since Z is the universal cover of the curve X and s is a section of $E(W)$, therefore one knows (cf. [NS]) that there exists a $\pi_1(X)$-invariant element $w \in W$ such that s is defined by a map $s_1 : Z \to W$, given by $s_1(x) = w$, $\forall x \in X$, i.e. “the constant map sending everything to w.”

Since $w \in W$ is a $\pi_1(X)$-invariant vector and the action of $\pi_1(X)$ is via the representation χ, we see that χ factors via

$$\chi_1 : \pi_1(X) \to H',$$

since $H' = \mathrm{Stab}_G(w)$.

In particular, we get the H'-bundle from the representation χ_1 and clearly this H'-bundle is the reduction of structure group of the G-bundle E given by the section s.

By the very construction of the reduction, the induced H'-bundle is *flat* and also semistable since it comes as the reduction of structure group of the polystable bundle E (by Definition 2.5). This proves the Proposition 5.2. □

5.1. Completion of proof of Theorem 5.1

By Lemma 4.1, we have

$$E_A\bigl(G_A/H'_A\bigr) \hookrightarrow E_A(W_A).$$

The given section s'_K of $E_K(G_K/H'_K)$ therefore gives a section u_K of $E(W_K)$. Further, $u_K(x)$, the restriction of u_K to $x \times (T - p)$, extends to give a section $u_A(x)$ of $E_x(W_A)$ (restriction of $E_A(W_A)$ to $x \times T$). Thus, by Proposition 2.8 and semistability of $E_p(W_A)$, the section u_K extends to give a section u_A of $E(W_A)$ over $X \times T$.

Now, to prove the Theorem 5.1, we need to make sure that

$$\text{The image of this extended section } u_A \text{ actually lands in } E_A(G_A/H'_A). \quad (*)$$

This would then define s'_A.

To prove $(*)$, it suffices to show that $u_A(X \times p)$ lies in $E_A(G_A/H'_A)_p$ (the restriction of $E_A(G_A/H'_A)$ to $X \times p$).

Observe that $u_A(x \times p)$ lies in $E_A(G_A/H'_A)_p$ since $u_A(x) = s'_A(x) = w_A$. Observe further that, if E_p denotes the principal G-bundle on X, which is the restriction of the G_A-bundle E_A on $X \times T$ to $X \times l$, then $E_A(W_A)_p = E_A(W_A)|X \times p$, and we also have

$$\begin{tikzcd} E_A(G_A/H'_A)_p \arrow[r, "\simeq"] \arrow[d] & E_p(G_k/H'_k) \arrow[d] \\ E_A(W_A)_p \arrow[r, "\simeq"] & E_p(W), \end{tikzcd}$$

where the vertical maps are inclusions:

$$E_A\big(G_A/H'_A\big)_p \hookrightarrow E_A(W_A)_p, \quad E_p\big(G_k/H'_k\big) \hookrightarrow E_p(W),$$

where $E_p(W) = E_p \times^{H'_k} W$ with fiber as the G-module $W = W_A \otimes k$. Note that G/H'_k is a G-subscheme Y of W.

Recall that E_p is polystable of degree zero. Then, from the foregoing discussion, the assertion that $u_A(X \times p)$ lies in $E_A(G_A/H'_A)$ is a consequence of Proposition 5.2 applied to E_p. (Note that the group $H'_k = \mathrm{Stab}_{G_k}(w_k)$ satisfies the hypothesis of Proposition 5.2.)

Thus we obtain a section s'_A of $E_A(G_A/H'_A)$ on $X \times T$, which extends the section s'_K of $E_A(G_A/H'_A)$ on $X \times (T - p)$. This gives a reduction of structure group of the G_A-bundle E_A on $X \times T$ to the subgroup scheme H'_A and this extends the given bundle E_K to the subgroup scheme H'_A.

In summary, we have extended the original H_K-bundle up to isomorphism to an H'_A-bundle. The extended H'_A-bundle has the property that the limiting bundle E'_p, which is an H'_k-bundle, comes with a reduction of structure group to the fundamental group of X and is *semistable* in the sense of Proposition 2.6. □

Remark 5.3. By Lemma 5.2, since the limiting bundle E'_p is polystable, we can conclude that the *monodromy subgroup* M' of E'_p, i.e. the minimal subgroup to which the structure group of E'_p can be reduced, is *reductive*, being the Zariski closure of the representation of the fundamental group of X defining the polystable E'_p.

Now recall the following *rigidity theorem* (cf. [SGA3, Corollary 2.8, III]), namely: since M' is reductive, the given inclusion $M' \hookrightarrow H'$ can be lifted to an inclusion of group schemes $M'_A \simeq M' \times \operatorname{Spec} A \hookrightarrow H'_A$ (possibly by going to a B which is integral over A). It follows then that M' can be embedded as a subgroup of H (recall that over the generic point of $\operatorname{Spec} A$ we have $H'_K \simeq H \times \operatorname{Spec} K$). Using this embedding we can thus extend structure group of E'_p to H!

It seems therefore that we have proved the semistable reduction theorem, for we have shown that the structure group G of E_p can be reduced to the subgroup $H \hookrightarrow G$. However, there is one crucial point to be proved, namely that all reductions vary *continuously*, in other words they fit together to give an H_A-bundle over $X \times T$. This is carried out in the next few sections with the aid of Bruhat–Tits theory.

6. Potential good reduction

To summarise, we have extended the original H_K-bundle up to isomorphism to an H'_A-bundle. To complete the proof of the Theorem 7.1, we need to extend the H_K-bundle to an H_A-bundle.

Remark 6.1. We note that in general the group scheme H'_A obtained above need not be a smooth group scheme over A. But in our case since the characteristic of the base field is zero and since H'_A is flat, it is also smooth over A.

Recall that H_A denotes the reductive group scheme $H \times \operatorname{Spec} A$ over A.

Proposition 6.2. *There exists a finite extension L/K with the following property: if B is the integral closure of A in L, and if H'_B are the pull-back group schemes, then we have a morphism of B-group schemes*

$$H'_B \to H_B,$$

which extends the isomorphism $H'_L \cong H_L$.

Proof. Observe first that the *lattice* $H'_A(A)$ is a *bounded subgroup* of $H_A(K)$, in the sense of the Bruhat–Tits theory [BT]. Here, we make the identifications:

$$H'_K \cong H_K \quad \text{as } K\text{-group schemes.}$$

Hence,

$$H'_A(A) \subset H'_K(K) \cong H_K(K) = H_A(K).$$

Then we use the following crucial fact:

$$\begin{aligned}&\text{There exists a finite extension } L/K \text{ and an element } g \in H'_A(L)\\ &\text{such that } g.H'_A(A).g^{-1} \hookrightarrow H_A(B).\end{aligned} \tag{$*$}$$

This assertion is a consequence of the following result from ([Se, Proposition 8, p. 546]) (cf. also [Gi, Lemma I.1.3.2], or [La, Lemma 2.4]):

> (Serre) There exists a totally ramified extension L/K having the following property: for every bounded subgroup M of $H(K)$, there exists $g \in H(K)$ such that $g.M.g^{-1}$ has *good reduction* in $H(L)$ (i.e. $h.M.h^{-1} \subset H(B)$, where B is the integral closure of A in L).

Larsen [La, (2.7), p. 619], concludes from $(*)$, in the l-adic case, the statement of Proposition 6.2. However, we give a complete proof.

For the sake of clarity we gather all the identifications of the subgroups under consideration:

$$H'_A(K) = H'_K(K), \qquad H'_A(L) = H'_B(L) = H'_L(L),$$
$$H'_A(A) \subset H'_B(B), \qquad H_A(B) = H_B(B).$$

Thus, we see that the isomorphism $\psi_L : H'_L \to H_L$, given by *conjugation by* g, induces a map $\psi_L(B) : H'_A(A) \to H_B(B)$. The crucial property to note is the following one:

> Given a rational point $\xi_k \in H'_k(k)$, there exists a point $\xi_A \in H'_A(A)$, and hence in $H'_B(B)$, which extends ξ_k, since H'_A is smooth over A and k is algebraically closed.

The proposition will follow by the following lemma. Let A, B, etc., be as above.

Lemma 6.3. *Let A be a complete discrete valuation ring with quotient field K. Let Z_A and Y_A be A-schemes with Z_A smooth. Let $\psi_L : Z_L \to Y_L$ be a L-morphism such that $\psi_L(B) : Z_A(A) \to Y_B(B)$. Then, the L-morphism ψ_K extends to a B-morphism $\psi_B : U_B \to Y_B$, where U_B is an open dense subscheme of Z_B which intersects all the irreducible components of the closed fiber Z_k.*

In particular, if Z_A and Y_A are smooth and separated group schemes and if ψ_L is a morphism of L-group schemes then there exists an extension $\psi_B : Z_B \to Y_B$ as a morphism of B-group schemes.

Proof. Consider the graph of ψ_L and denote its schematic closure in $Z_B \times_B Y_B$ by Γ_B. Let $p : \Gamma_B \to Z_B$ be the first projection. Then p is an isomorphism on generic fibers. So, it is enough if we prove that p is invertible on an open dense B-subscheme U_B of Z_B, which intersects all the components C, of the closed fiber Z_k.

We claim that the map $p_k : \Gamma_k \to Z_k$ is surjective onto the subset of k-rational points of each components, and this will imply that p_k is surjective since k is algebraically closed. Note that Z_A is assumed to be smooth and so, the closed fiber is reduced and also k is algebraically closed. Thus, each $z_k \in Z_k(k)$ lifts to a point $z \in Z_A(A) \subset Z_B(B)$, A, being a complete discrete valuation ring. Since $\psi_L(B)$ maps $Z_A(A) \to Y_B(B)$, we see that, there exists a $y \in Y_B(B)$ such that $(z, y) \in \Gamma_B(B)$. Thus, z_k lies in the image of p_k. This proves the claim.

In particular, the generic points, α's, of all the components C, of Z_k (by Chevalley), lie in the image of p_k. Let $p_k(\xi) = \alpha$. Consider the local rings $\mathcal{O}_{\Gamma_B,\xi}$ and $\mathcal{O}_{Z_B,\alpha}$. Then, by the above claim, the local ring $\mathcal{O}_{\Gamma_B,\xi}$ dominates $\mathcal{O}_{Z_B,\alpha}$. Since Z_B is smooth and hence normal, for every α the local rings $\mathcal{O}_{Z_B,\alpha}$ are all discrete valuation rings. Further, since Γ_B is the schematic closure of Γ_L, it implies that Γ_B is B-flat and Γ_L is open and dense in Γ_B. Moreover, since p is an isomorphism on generic fibers, both local rings have the same quotient rings. Finally, since $\mathcal{O}_{Z_B,\alpha}$ is a discrete valuation ring, we have an isomorphism of local rings. Thus, the schemes being of finite type over B, we have open subsets $V_{i,B}$ and $U_{i,B}$ for each component of Z_k, which we index by i, such that p induces an isomorphism between $V_{i,B}$ and $U_{i,B}$. This gives an extension of ψ to open subsets $U_{i,B}$ for every i, with the property that these maps agree on the generic fiber. Since Y_B is separated, these extensions glue together to give an extension ψ_B on an open subset, which we denote by U_B; this open subset will of course intersect all the components of the closed fibers of Z_k.

The second part of the lemma follows immediately if Y_A is affine (which is our case). More generally, we appeal to the general theorem of A. Weil on morphisms into group schemes, which says that if a rational map ψ_B is defined in codimension $\leqslant 1$ and if the target space is a group scheme then it extends to a global morphism. (cf., for example, [BLR, p. 109]). As we have checked above, this holds in our case and implies that as a morphism of schemes ψ_L extends to give $\psi_B : Z_B \to Y_B$.

Further, by assumption, ψ_L is already a morphism of L-group schemes, and hence it is easy to see that the extension ψ_B is also a morphism of B-group schemes. This concludes the proof of the lemma. □

7. Semistable reduction theorem

The aim of this section is to prove the following theorem.

Theorem 7.1. *Let P_K be a family of semistable principal H-bundles on $X \times$ Spec K, or equivalently, if H_K denotes the group scheme $H \times$ Spec K, a semistable H_K-bundle P_K on X_K. Then there exists a finite extension L/K, with the integral closure B of A in L such that P_K, after base change to* Spec B, *extends to a semistable H_B-bundle P_B on X_B.*

Remark 7.2. Let $H \subset G$, where G is a linear group. In the notation of Section 2, let F_H and F_G stand for the functors associated to families of semistable bundles of degree zero (cf. Proposition 2.8). The inclusion of H in G induces a morphism of functors $F_H \to F_G$. We remark that the semistable reduction theorem for principal H-bundles *need not* imply that the induced morphism $F_H \to F_G$ is a proper morphism of functors. Indeed, this does not seem to be the case. However, it does imply that the associated morphism at the level of moduli spaces is indeed proper (cf. Theorem 8.5).

7.1. Completion of the proof of the Theorem 7.1

Thus, in conclusion, first by Proposition 5.1 we have an H'_A-bundle which extends the H_K-bundle up to isomorphism. Then, by Proposition 6.2, going to the extension L/K, we have a morphism of B-group schemes $\psi_B : H'_B \to H_B$, which is an isomorphism over L. Therefore, one can extend the structure group of the bundle E'_B to obtain an H_B-bundle E_B which extends the H_K-bundle E_K.

Moreover, the fiber of E_B over the closed point is indeed *semistable*.

To see this, observe first that it comes as the extension of structure group of E'_p by the map $\psi_k : H'_k \to H_k$. Recall (Proposition 5.2) that E'_p is the *semistable* H'_k-bundle obtained as the reduction of structure group of the polystable vector bundle $E(V_A)_p$ and so remains semistable by any associated construction (cf. Proposition 2.6).

This completes the proof of the Theorem 7.1. $\square$

8. Construction of the moduli space

For the present purpose, we take $G = SL(n, \mathbb{C})$ and $H \subset G$ a semisimple subgroup.

We recall very briefly the Grothendieck Quot scheme used in the construction of the moduli space of vector bundles (cf. [Ses]).

Let $\mathcal{F}$ be a coherent sheaf on X and let $\mathcal{F}(m)$ be $\mathcal{F} \otimes \mathcal{O}_X(m)$ (following the usual notations). Choose an integer $m_0 = m_0(n, d)$ (n is a rank, d is a degree) such that for any $m \geqslant m_0$ and any semistable bundle V of rank n and degree d on X and we have $h^i(V(m)) = 0$ and $V(m)$ is generated by its global sections.

Let $\chi = h^0(V(m))$ and consider the Quot scheme Q consisting of coherent sheaves $\mathcal{F}$ on X which are quotients of $\mathbb{C}^\chi \otimes_{\mathbb{C}} \mathcal{O}_X$ with a fixed Hilbert polynomial P. The group $\mathcal{G} = GL(\chi, \mathbb{C})$ canonically acts on Q and hence on $X \times Q$ (trivial action on X) and lifts to an action on the universal sheaf $\mathcal{E}$ on $X \times Q$.

Let R denote the $\mathcal{G}$-invariant open subset of Q defined by

$$R = \{ q \in Q \mid \mathcal{E}_q = \mathcal{E}|_{X \times q} \text{ is locally free such that the canonical map } \mathbb{C}^\chi \to H^0(\mathcal{E}_q) \text{ is an isomorphism, } \det \mathcal{E}_q \simeq \mathcal{O}_X \}.$$

We denote by Q^{ss} the $\mathcal{G}$-invariant open subset of R consisting of semistable bundles and let $\mathcal{E}$ continue to denote the restriction of $\mathcal{E}$ to $X \times Q^{ss}$.

Henceforth, 'by abuse of notation,' we shall write Q for Q^{ss}.

Proof of Lemma 3.1. Note that the moduli space in question, namely, of G-semistable bundles, is a GIT quotient $Q \to M$ by $\mathcal{G}$, and the family $E_A(G)$ is given by a morphism $T \to M$. Lift the K-valued point, namely, r_K, given by the family E_K, to Q and consider the $\mathcal{G}$-orbit R_0 of r_K in Q. Let $\overline{R}_0$ be its closure in Q. Since the K-valued point r_K is in fact an A-valued point of M, the GIT quotient of $\overline{R}_0$ is indeed the curve T. Also, observe that the closure intersects the closed fiber. Consider the morphism $\psi : \overline{R}_0 \to T$. Since the base is a curve T, one has a *multi-section* for the morphism ψ, and one obtains the curve S. The general fiber has been modified only in the orbit, therefore, the isomorphism class of the bundles remains unchanged. □

8.1. The construction of the moduli space for principal bundles

Fix a base point $x \in X$ (cf. Remark 2.3). Let $q'' : (\mathrm{Sch}) \to (\mathrm{Sets})$ be the following functor:

$$q''(T) = \{ (V_t, s_t) \mid \{V_t\} \text{ is a family of semistable principal } G\text{-bundles parameterised by } T \text{ and } s_t \in \Gamma(X, V(G/H)_t)\ \forall t \in T \},$$

i.e. $q''(T)$ consists pairs of rank-n vector bundles (or equivalently principal G-bundles) together with a reduction of structure group to H.

By appealing to the general theory of Hilbert schemes, one can show without much difficulty (cf. [R1, Lemma 3.8.1]) that q'' is representable by a Q-scheme, which we denote by Q''.

The universal sheaf $\mathcal{E}$ on $X \times Q$ is in fact a vector bundle. Denoting by the same $\mathcal{E}$ the associated principal G-bundle, set $Q' = (\mathcal{E}/H)_x$. Then in our notation $Q' = \mathcal{E}(G/H)_x$; i.e., we take the bundle over $X \times Q$ associated to $\mathcal{E}$ with fiber G/H and take its restriction to $x \times Q \approx Q$. Let $f : Q' \to Q$ be the natural map. Then, since H is reductive, f is an *affine morphism*.

Observe that Q' parameterizes semistable vector bundles together with *initial values at x of possible reductions to H*.

Define the "evaluation map" of Q-schemes as follows:

$$\phi_x : Q'' \to Q', \quad (V, s) \longmapsto (V, s(x)).$$

Lemma 8.1. *The evaluation map $\phi_x : Q'' \to Q'$ is* proper.

Proof. The lemma follows easily from the proof of Proposition 2.8. □

Lemma 8.2. *The evaluation map ϕ_x is* injective.

Proof. Let $G/H \hookrightarrow W$ be as in Proposition 2.8 and let (E, s) and $(E', s') \in Q''$ such that $\phi_x(E, s) = \phi_x(E', s')$ in Q', i.e., $(E, s(x)) = (E', s'(x))$. So we may assume that $E \simeq E'$ and that s and s' are two different sections of $E(G/H)$ with $s(x) = s'(x)$.

Using $G/H \hookrightarrow W$, we may consider s and s' as sections in $\Gamma(X, E(W))$. Observe that, by definition, E being semistable of degree 0, so is $E(W)$.

Recall the following fact:

If E and F are semistable vector bundles with $\mu(E) = \mu(F)$, then the evaluation map

$$\phi_x : \mathrm{Hom}(E, F) \to \mathrm{Hom}(E_x, F_x) \tag{$*$}$$

is *injective*.

In our situation, s, and $s' \in \mathrm{Hom}(\mathcal{O}_X, E(W))$ and hence by $(*)$, since $\phi_x(s) = \phi_x(s')$, we get $s = s'$, proving injectivity. $\square$

Remark 8.3. It is immediate that the $\mathcal{G}$-action on Q lifts to an action on Q''.

Recall the commutative diagram

$$\begin{tikzcd} Q'' \arrow[r, "\phi_x"] \arrow[dr, "\psi"'] & Q' \arrow[d, "f"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Q \end{tikzcd}$$

By Lemmas 8.1 and 8.2, ϕ_x is a proper injection and hence affine. One knows that f is affine (with fibers G/H). Hence ψ is a $\mathcal{G}$ *equivariant affine morphism.*

Lemma 8.4. *Let (E, s) and (E', s') be in the same $\mathcal{G}$-orbit of Q''. Then we have $E \simeq E'$. Identifying E' with E, we see that s and s' lie in the same orbit of* $\mathrm{Aut}_G\, E$ *on $\Gamma(X, E(G/H))$. Then using Remark* 2.1(b)*, we see that the reductions s and s' give isomorphic H-bundles.*

Conversely, if (E, s) and (E', s') are such that $E \simeq E'$ and the reductions s, s' give isomorphic H-bundles, again using Remark 2.1(b)*, we see that (E, s) and (E', s') lie in the same $\mathcal{G}$-orbit.*

Consider the $\mathcal{G}$-action on Q'' with the linearization induced by the *affine $\mathcal{G}$-morphism* $Q'' \to Q$. It is seen without much difficulty that, since a good quotient of Q by $\mathcal{G}$ exists and since $Q'' \to Q$ is an affine $\mathcal{G}$-equivariant map, a good quotient $Q''/\mathcal{G}$ exists (cf. [R1, Lemma 4.1]). Moreover, by the universal property of categorical quotients, the canonical morphism

$$\overline{\psi} : Q''//\mathcal{G} \to Q//\mathcal{G}$$

is also *affine*.

Theorem 8.5. *Let $M_X(H)$ denote the scheme $Q''//\mathcal{G}$. Then this scheme is the coarse moduli scheme of semistable H-bundles. Further, $M_X(H)$ is projective and if $H \hookrightarrow GL(V)$ is a faithful representation, the canonical morphism $\overline{\psi} : M_X(H) \to M_X(GL(V))$ is* finite.

Proof. We need only check the last statement. By Theorem 3.1 one sees easily that the moduli space $M_X(H)$ is projective, and therefore $\overline{\psi}$ is *proper*. By the remarks above $\overline{\psi}$ is also *affine*, therefore it follows that $\overline{\psi}$ is *finite*. □

Remark 8.6. We have supposed that H is semisimple; however, it is not difficult to treat the more general case when H is *reductive*. Let H be reductive and $\overline{H} = H$ mod centre, its adjoint group. Let P be a principal H-bundle and $\overline{P}$ the $\overline{H}$-bundle, obtained by extension of structure groups. *We define P to be semistable if $\overline{P}$ is semistable*. If we fix a topological isomorphism class c for principal H-bundles, this fixes a topological isomorphism $\bar{c}$ for principal $\overline{H}$-bundles. Then the moduli space $M_X(H)_c$ is "essentially" $M_X(\overline{H})_{\bar{c}} \times$ (product of Jacobians). This can be made rigorous and it leads to the construction of $M_X(H)_c$.

8.2. Points of the moduli space

In this subsection we will briefly describe the k-valued points of the moduli space $M_X(H)$. The general functorial description of $M_X(H)$ as a coarse moduli scheme follows by the usual process.

Recall the following definitions from [R1].

Definition 8.7 (A. Ramanathan). E is *semistable* if for any parabolic subgroup P of H, any reduction $\sigma_P : X \to E(H/P)$ and any dominant character χ of P, the bundle $\sigma_P^*(L_\chi))$ has degree $\leqslant 0$ (cf. [R1]). We note that in this convention, a dominant character χ of P induces a negative ample line bundle on G/P.

Note that this definition makes sense for reductive groups as well.

Definition 8.8. A reduction of structure group of E to a parabolic subgroup P is called *admissible* if for any character χ on P, which is trivial on the center of H, the line bundle associated to the P-bundle E_P, obtained by the reduction of structure group, has degree zero.

Definition 8.9. An H-bundle E is said to be *polystable* if it has a reduction of structure group to a Levi subgroup R of a parabolic P such that the R-bundle E_R, obtained by the reduction, is stable and the extended P-bundle $E_R(P)$ is an admissible reduction of structure group for E.

Proposition 8.10. *The "points" of $M_X(H)$ are given by isomorphism classes of polystable principal H-bundles.*

We first remark that, since the quotient $q : Q'' \to M_X(H)$ obtained above is a good quotient, it follows that each fiber $q^{-1}(E)$ for $E \in M_X(H)$ has the unique closed $\mathcal{G}$-orbit. Let us denote the orbit $\mathcal{G} \cdot E$ by $O(E)$. The proposition will follow from the following lemma.

Lemma 8.11. *If $O(E)$ is closed then E is polystable.*

Proof. Recall the definition of a polystable bundle (Definition 8.9) and the definition of *admissible reductions* (Definition 8.8). If E has no admissible reduction of structure group to a parabolic subgroup then it is polystable, and there is nothing to prove.

Suppose then that E has an admissible reduction E_P to $P \subset H$. Recall by the general theory of parabolic subgroups that there exists a 1-PS $\xi : \boldsymbol{G}_m \to H$ such that $P = P(\xi)$. Let $L(\xi)$ and $U(\xi)$ be its canonical Levi subgroup and unipotent subgroup, respectively. The Levi subgroup will be the centralizer of this 1-PS ξ and one knows $P(\xi) = L(\xi) \cdot U(\xi) = U(\xi) \cdot L(\xi)$. In particular, if $h \in P$ then $\lim \xi(t) \cdot h \cdot \xi(t)^{-1}$ exists. From these considerations one can show that there is a morphism

$$f : P(\xi) \times \boldsymbol{A}^1 \to P(\xi)$$

such that $f(h, 0) = m \cdot u$, where $h \in P$ and $h = m \cdot u$, $m \in L$ and $u \in U$ (see [R1, Lemma 3.5.12]).

Consider the P-bundle E_P. Then, using the natural projection $P \to L$ where $L = L(\xi)$, we obtain an L-bundle $E_P(L)$. Again, using the inclusion $L \hookrightarrow P \hookrightarrow H$, we obtain a new H-bundle $E_P(L)(H)$. Let us denote this H-bundle by $E_P(L, H)$. It follows from the definition of admissible reductions and polystability that $E_P(L, H)$ is *polystable*.

Further, from the family of maps f defined above, composing them with the inclusion $P(\xi) \hookrightarrow H$, we obtain a family of H-bundles $E_P(f_t)$ for $t \neq 0$; all these bundle are isomorphic to the given bundle E. Following [R1, Proposition 3.5, p. 313], one can prove that the bundle $E_P(L, H)$ is the limit of $E_P(f_t)$. It follows that $E_P(L, H)$ is in the $\mathcal{G}$-orbit $O(E)$ because $O(E)$ is closed. Now, by Lemma 8.4, $E \simeq E_P(L, H)$, implying that E is polystable. □

9. Semistable reduction for principal Higgs bundles

The aim of this section is to extend the methods of Section 3 and to prove the analogue of Theorem 7.1 for the case of principal Higgs bundles (cf. Theorem 9.3).

9.1. Higgs vector bundles

We recall briefly the usual category of *semistable* Higgs bundles with vanishing Chern classes or what are called by Simpson *semiharmonic bundles* (for details, cf. [Sim0, p. 49]).

Suppose that X is a smooth projective variety over $k = \mathbb{C}$ and we have fix a polarization to enable us to define *degree* of bundles. A *Higgs bundle* is a holomorphic vector bundle E together with a holomorphic map $\theta : E \to E \otimes \Omega_X^1$ such that $\theta \wedge \theta = 0$ in $\mathrm{End}(E) \otimes \Omega_X^2$. Define a Higgs bundle E to be *semistable* (respectively *stable*) if for every non-zero subsheaf $V \subset E$ preserved by θ,

$$\frac{\deg V}{\mathrm{rk}\, V} \leqslant \frac{\deg E}{\mathrm{rk}\, E} \quad (\text{respectively } <),$$

where we choose a hyperplane class h and define the degree as $c_1(E).[h]^{n-1}$. Let us say that a Higgs bundle E is *polystable* if it is the direct sum of stable Higgs bundles of the same slope (where slope is defined as usually as deg / rk). Following Simpson, we call a semistable Higgs bundle *of semiharmonic type* if all its Chern classes are zero.

If E is a Higgs bundle, define the *space of Higgs sections* or $H^0_{\mathrm{dol}}(X, E)$ to be the space of holomorphic sections s such that $\theta.s = 0$.

- Then, one has the basic theorem on Higgs bundles which says that there is an equivalence of categories between the category of polystable Higgs bundles of rank n which are *semiharmonic* and the category of semisimple representations of $\pi_1(X) \to GL(n)$.
- From this one can deduce as in Section 2, the *tensor product theorem* for semistable Higgs bundles. This in particular implies that, if $\rho : GL(n) \to GL(W)$ is a finite dimensional representation and if (E, θ) is a semistable Higgs bundle with vanishing Chern classes, then the associated bundle $E(W)$ is also semistable with the induced Higgs structure (with vanishing Chern classes).

Let (E, θ) be a Higgs bundle. Then we consider the characteristic polynomial of Higgs structure θ with its coefficients as points in the space

$$\bigoplus_{1 \leq i \leq n} H^0\big(X, \mathrm{Sym}^i\, \Omega_X^1\big).$$

We define this element in the above direct sum as the *characteristic tuple of* (E, θ). Then one has the following basic theorem due to N. Hitchin, N. Nitsure, and C. Simpson.

Theorem 9.1. *There is a quasi projective variety M_{Higgs} whose points parameterize polystable Higgs bundles on X with vanishing Chern classes. The map from M_{Higgs} to the space of polynomials with coefficients in the symmetric powers of the cotangent bundle, taking any (E, θ) to its characteristic tuple is proper.*

We have the following proposition similar to Lemma 3.1.

Proposition 9.2. *Let $T = \mathrm{Spec}(A)$ where A is a discrete valuation ring with residue field $k = \mathbb{C}$ and quotient field K. Let E_K be a family of semistable Higgs bundles on $X_K = X \times \mathrm{Spec}\, K$ with vanishing Chern classes and fixed characteristic tuple. Then, by going to a finite cover if necessary, there exists a family E_A which extends E_K to X_A so that the fiber E_p over the closed point $p \in \mathrm{Spec}\, A$ is polystable with vanishing Chern classes.*

9.2. Semistable principal Higgs bundles

Following Simpson [Sim2, Section 9], we define a *principal Higgs bundle* on X for a reductive algebraic group H with Lie algebra $\boldsymbol{h}$, is a principal H-bundle $E \to X$ together with a section θ of $(E \times^H \boldsymbol{h}) \otimes \Omega^1_X$ such that $\theta \wedge \theta = 0$ in $(E \times^H \boldsymbol{h}) \otimes \Omega^2_X$. Given such an object and a representation $H \to GL(V)$, we get a Higgs bundle $E(V) = E \times^H V$.

Say that E is *semistable* if the Chern classes of E are all zero and if for a faithful representation $H \hookrightarrow GL(V)$ the associated bundle $E(V)$ is a semistable Higgs bundle. As has been noted above, by the tensor product theorem, this is independent of the choice of the representation. Note that this definition can be naturally relativized for a variety over T. With this definition our aim is then to prove the following theorem.

Theorem 9.3. *Suppose that we are given a family of semistable principal Higgs H-bundle (E_K, θ_K) on $X \times \mathrm{Spec}\, K$, or equivalently, if H_K denotes the group scheme $H \times \mathrm{Spec}\, K$, we are given a semistable Higgs H_K-bundle E_K on X_K. Suppose further that, for the associated Higgs vector bundle family $(E(V)_K, \theta(V)_K)$, the characteristic tuple is fixed. Then there exists a finite extension L/K with the integral closure B of A in L such that E_K, when pulled back to $\mathrm{Spec}\, B$, extends to a semistable Higgs H_B-bundle E_B on X_B.*

9.3. Higgs section

We have the following lemma which is necessary to prove the result corresponding to Proposition 2.8 or Lemma 2.10. (For a related result, cf. [Sim2, Theorem 9.6].)

Proposition 9.4. *Let $T = \operatorname{Spec} A$ and let (E_T, θ_T) be family of Higgs bundles with vanishing Chern classes and fixed characteristic. Let $s_K : \mathcal{O}_{X_K} \to E_K$ be a family of Higgs sections such that for a base point $x \in X$ the section extends along $x \times T$. Then the family s_K extends to $s_T : \mathcal{O}_{X_T} \to E_T$, to whole T.*

Proof. The proof of this proposition is much the same as the proof of Proposition 2.8. The only new ingredient needed to complete it is the following fact about Higgs bundles.

Lemma 9.5. *Let V and W be semistable Higgs bundles with vanishing Chern classes. Let $x \in X$ be a base point. The we have an injection*

$$\mathrm{Hom}_{\mathrm{Higgs}}(V, W) \hookrightarrow \mathrm{Hom}(V_x, W_x),$$

or equivalently, a non-zero Higgs section is nowhere zero.

Proof. (Cf. [Sim1, Lemma 4.9].) □

9.4. Monodromy subgroups, polystability and local constancy

Following [Sim2, Theorem 9.8] we define *the monodromy subgroup* of the polystable Higgs bundle E as *a* subgroup M which is minimal among all subgroups of G with the property that the structure group of E can be reduced to M and such that the reduced bundle E_M is *semiharmonic* (note that as defined M is not unique as we have not fixed a base point in the fiber of E at $x \in X$ as Simpson [Sim0, p. 29]).

We need the following *local constancy* property which we isolate in a proposition (cf. also [Sim0, Lemma 2.10]).

Proposition 9.6. *Let (E, θ) be a* polystable *principal Higgs G-bundle on X with vanishing Chern classes* (*or* semiharmonic) (*here $G = SL(n)$ or $GL(n)$*). *Let W be a G-module and Y a G-subscheme of W of the form G/H', where $H' = \mathrm{Stab}_G(w)$ for some $w \in W$.*

Then if s is a Higgs section of $E(W)$ such that for some $x \in X$, $s(x) = w$ is in the fiber of $E(Y)$ at $x \in X$, then the entire image of s lies in $E(Y)$. In particular, the principal G-bundle has a reduction of structure group to H'. Furthermore, the reduced H'-bundle is also semiharmonic in the sense of [Sim2].

Proof. The proof follows the proof of Proposition 5.2. The section s being a non-zero Higgs section, is nowhere zero by Lemma 9.5. By [Sim2, Theorem 9.8], since E is polystable, $E(W)$ is also polystable. Therefore the section s of $E(W)$ gives a splitting of $E(W)$ as

$$E \simeq \bigoplus V_i \oplus \mathcal{O}_X.$$

This gives a reduction of the structure group of $E(W)$ to a group which is the Levi subgroup L of a maximal parabolic subgroup of $GL(W)$ corresponding to the extension of $E(W)$ as $\mathcal{O}_X$ by $\bigoplus V_i$, V_i being stable Higgs bundles.

Thus, the bundle $E(W)$ has two *semiharmonic* reductions of structure group, namely, to the subgroups G and L of $GL(W)$. Therefore, by the definition of the monodromy subgroup of $E(W)$, we have $M \hookrightarrow G \cap L$. Let E_M be the reduced M-bundle. The section s which gives the copy of $\mathcal{O}_X$ in $E(W)$ can therefore be thought of as a section of $E_M(W)$ obtained by the *trivial character* on M.

Since the value of the section is given at $x \in X$, namely $s(x) = w$, the section of $E(W) = E_M(W)$ can be seen as obtained by the *constant map* $E_M \to W$ which maps the whole of E_M to an M-invariant vector $w \in W$ (cf. proof of Proposition 5.2).

Exactly as in the proof of Proposition 5.2, we see that the inclusion $M \hookrightarrow G$ factors via an inclusion $M \hookrightarrow H'$ since $H' = \mathrm{Stab}_G(w)$.

Now, taking $E'_H = E_M(H')$, we get the required reduction of structure group of E to H'.

Again, since E_M, by the definition of monodromy subgroup, is *semiharmonic*, it follows that the induced H'-bundle is also *semiharmonic* proving the proposition. □

Remark 9.7. In the proof of Proposition 5.2 instead of the monodromy reduction we realize the bundle as extension of structure group from the principal $\pi_1(X)$-bundle $j : Z \to X$, the universal covering space of X. Notice that the monodromy subgroup of (E, θ) can be identified with the Zariski closure of the monodromy representation giving the polystable Higgs bundle (E, θ).

9.5. *Extension to the flat closure and potential good reduction*

Once this proposition is proven, then we follow the strategy of the proof of Theorem 7.1. We can extend the family (E_K, θ_K) to a family of Higgs bundles with structure group scheme H'_A, the *flat closure* of H_K in $GL(V_A)$. Here we can define the notion of a Higgs bundle for a non-reductive group simply as a principal bundle which becomes Higgs semistable for a faithful representation, etc. (in fact, Simpson does not assume his group is reductive to define the notion of a semistable principal Higgs bundle).

The rest of the proof is verbatim from Section 3, Proposition 6.2, and we have Theorem 9.3.

Remark 9.8. One can proceed as in Section 4 and obtain a construction of the moduli space $M_{\mathrm{Higgs}}(H)$ of semistable Higgs H-bundles with vanishing Chern classes, and, as a consequence, in much the same way one can prove, as in Theorem 8.5, that the natural morphism $M_{\mathrm{Higgs}}(H) \to M_{\mathrm{Higgs}}(G)$, induced from a representation $H \to G$, is *finite* (cf. [Sim2, Section 9]).

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